Gaussian fluctuations in chaotic eigenstates

1996 J. Phys. A: Math. Gen. 29 5817

(http://iopscience.iop.org/0305-4470/29/18/013)

J. Phys. A: Math. Gen. **29** (1996) 5817–5826. Printed in the UK

Gaussian fluctuations in chaotic eigenstates

Mark Srednicki† and Frank Stiernelof
Department of Physics, University of California, Santa Barbara, CA 93106, USA

Received 10 April 1996

Abstract. We study the fluctuations that are predicted in the autocorrelation function of an energy eigenstate of a chaotic, two-dimensional billiard by the conjecture (due to Berry) that the eigenfunction is a Gaussian random variable. We find an explicit formula for the root-mean-square amplitude of the expected fluctuations in the autocorrelation function. These fluctuations turn out to be $O(\hbar^{1/2})$ in the small $\hbar$ (high energy) limit. For comparison, any corrections due to scars from isolated periodic orbits would also be $O(\hbar^{1/2})$. The fluctuations take on a particularly simple form if the autocorrelation function is averaged over the direction of the separation vector. We compare our various predictions with recent numerical computations of Li and Robnik for the Robnik billiard, and find good agreement. We indicate how our results generalize to higher dimensions.

Two-dimensional billiards which are classically chaotic have proven to be an efficient laboratory for the study of quantum chaos. The energy eigenvalues and eigenfunctions can be computed with good accuracy, and compared with theoretical predictions of their properties. These predictions are typically semiclassical in nature, involving properties that are expected to be emergent in the formal limit of $\hbar \to 0$.

In practice, numerical methods are used to find the eigenvalues k^2 and eigenfunctions $\psi_k(\boldsymbol{x})$ of the time-independent Schrödinger equation,

$$(\nabla^2 + k^2)\psi_k(\boldsymbol{x}) = 0 \tag{1}$$

where $\boldsymbol{x}$ is in the domain B of the billiard, and the Dirichlet boundary condition

$$\psi_k(\boldsymbol{x}) = 0 \qquad \text{on } \partial B \tag{2}$$

is imposed. Then large k corresponds to small $\hbar$; an expansion of some quantity in powers of the wavelength $\lambda = 2\pi/k$ corresponds to an expansion in powers of $\hbar$.

Our focus here will be on the autocorrelation function $C_{k,R}(\boldsymbol{s})$, introduced by Berry [1]. Given an eigenfunction $\psi_k(\boldsymbol{x})$, a separation vector $\boldsymbol{s}$, and an averaging region R, the autocorrelation function is defined to be

$$C_{k,R}(\boldsymbol{s}) \equiv \frac{A_B}{A_R}\int_R \mathrm{d}^2x\; \psi_k(\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{s})\psi_k(\boldsymbol{x} - \tfrac{1}{2}\boldsymbol{s}) \tag{3}$$

where A_B is the area of the billiard and A_R is the area of the averaging region R. The eigenfunction $\psi_k(\boldsymbol{x})$ is assumed to be real, and normalized in the usual way,

$$\int_B \mathrm{d}^2x\; \psi_k^2(\boldsymbol{x}) = 1. \tag{4}$$

† E-mail address: mark@tpau.physics.ucsb.edu

0305-4470/96/185817+10$19.50

However, our normalization of $C_{k,R}(\boldsymbol{s})$ differs slightly from Berry's; we will discuss the reason for this later.

Berry conjectured that an energy eigenfunction in a chaotic billiard would appear locally to be a superposition of plane waves with random directions of the momenta and random phases. This implies that the expected value of $C_{k,R}(\boldsymbol{s})$ is

$$\begin{aligned}\langle C_{k,R}(\boldsymbol{s})\rangle &= \frac{\int \mathrm{d}^2p\,\delta(\boldsymbol{p}^2-k^2)\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{s}}}{\int \mathrm{d}^2p\,\delta(\boldsymbol{p}^2-k^2)} \\ &= J_0(ks)\end{aligned} \tag{5}$$

where $s = |\boldsymbol{s}|$, and $J_0(x)$ is a Bessel function. We now know that there are corrections to this result ('scars' [2]) associated with isolated periodic orbits in the classical billiard. Assuming that the averaging region R encompasses many wavelengths in the direction perpendicular to each orbit giving a scar, then the scar corrections to (5) are suppressed by $\mathrm{O}(\hbar^{1/2})$.

In the limit that the number of superposed plane waves becomes infinite, the central limit theorem tells us that the function $\psi_k(\boldsymbol{x})$ can be treated as a Gaussian random variable [1, 3, 4]. This means that we have prior information (in the sense used in Bayesian statistical analysis) about $\psi_k(\boldsymbol{x})$ which can be represented by a functional probability distribution of the form

$$P(\psi|k) = \mathcal{N}\exp\left[-\tfrac{1}{2}\int_B \mathrm{d}^2x_1\int_B \mathrm{d}^2x_2\,\psi(\boldsymbol{x}_1)K(\boldsymbol{x}_1,\boldsymbol{x}_2;k)\psi(\boldsymbol{x}_2)\right]. \tag{6}$$

Here $\mathcal{N}$ is a normalization constant, and $K(\boldsymbol{x}_1,\boldsymbol{x}_2;k)$ is the inverse of $A_B^{-1}J_0(k|\boldsymbol{x}_1-\boldsymbol{x}_2|)$ in the sense that

$$\int_B \mathrm{d}^2x_2\,K(\boldsymbol{x}_1,\boldsymbol{x}_2;k)J_0(k|\boldsymbol{x}_2-\boldsymbol{x}_3|) = A_B\delta^2(\boldsymbol{x}_1-\boldsymbol{x}_3) \tag{7}$$

with $\boldsymbol{x}_1$ and $\boldsymbol{x}_3$ restricted to B. Given an eigenvalue k^2, $P(\psi|k)\prod_{\boldsymbol{x}\in B}\mathrm{d}\psi(\boldsymbol{x})$ is the probability that the actual eigenfunction $\psi_k(\boldsymbol{x})$ is between $\psi(\boldsymbol{x})$ and $\psi(\boldsymbol{x})+\mathrm{d}\psi(\boldsymbol{x})$ for all points $\boldsymbol{x}$ in the billiard.

Angle brackets around some quantity depending on $\psi_k(\boldsymbol{x})$ are now defined to represent an average of that quantity over the probability distribution $P(\psi_k|k)$. Thus we have

$$\langle\psi_k(\boldsymbol{x}_1)\psi_k(\boldsymbol{x}_2)\rangle = A_B^{-1}J_0(k|\boldsymbol{x}_1-\boldsymbol{x}_2|). \tag{8}$$

Combining this with the definition (3) of $C_{k,R}(\boldsymbol{s})$ gives $\langle C_{k,R}(\boldsymbol{s})\rangle = J_0(ks)$, in agreement with (5). However, the probability distribution (6) also contains information about the fluctuations of $C_{k,R}(\boldsymbol{s})$ about $\langle C_{k,R}(\boldsymbol{s})\rangle$. Our goal is to study the properties of these fluctuations.

Before proceeding, let us recall that there is striking numerical evidence in favour of another consequence of (6): specifically, the probability $P(\chi)\,\mathrm{d}\chi$ that any actual eigenfunction has a value between χ and $\chi+\mathrm{d}\chi$ at any particular point $\boldsymbol{x}$ is given by

$$P(\chi) = (A_B/2\pi)^{1/2}\exp(-\tfrac{1}{2}A_B\chi^2). \tag{9}$$

This prediction can be tested by dividing the billiard into small pixels, and making a histogram of the value of the eigenfunction at each pixel. This was first done by MacDonald and Kaufman [4] in their study of eigenfunctions of the stadium billiard with $k^2A_B \simeq 1.3\times10^4$. More recently, Li and Robnik [5] studied eigenfunctions of the Robnik billiard [6] with $k^2A_B \simeq 2.5\times10^6$, and found excellent agreement with (9). Generally, the prediction (6) is expected to be valid provided that distortions of the billiard boundary on the scale of the wavelength $\lambda = 2\pi/k$ do not permit the formation of an integrable billiard [4].

Our main tool in studying the fluctuations of $C_{k,R}(\boldsymbol{s})$ about $\langle C_{k,R}(\boldsymbol{s})\rangle$ will be the relation [3, 7, 8]

$$\begin{aligned}\langle\psi_k(\boldsymbol{x}_1)\psi_k(\boldsymbol{x}_2)\psi_k(\boldsymbol{x}_3)\psi_k(\boldsymbol{x}_4)\rangle &= \langle\psi_k(\boldsymbol{x}_1)\psi_k(\boldsymbol{x}_2)\rangle\langle\psi_k(\boldsymbol{x}_3)\psi_k(\boldsymbol{x}_4)\rangle \\ &\quad+\langle\psi_k(\boldsymbol{x}_1)\psi_k(\boldsymbol{x}_3)\rangle\langle\psi_k(\boldsymbol{x}_2)\psi_k(\boldsymbol{x}_4)\rangle + \langle\psi_k(\boldsymbol{x}_1)\psi_k(\boldsymbol{x}_4)\rangle\langle\psi_k(\boldsymbol{x}_2)\psi_k(\boldsymbol{x}_3)\rangle. \qquad (10)\end{aligned}$$

The derivation of (10) from (6) can be found in textbooks on Euclidean quantum field theory (e.g. [9]). We will focus on the quantity

$$\begin{aligned}\Delta_{k,R}(\boldsymbol{s}_1,\boldsymbol{s}_2) &\equiv \langle[C_{k,R}(\boldsymbol{s}_1) - \langle C_{k,R}(\boldsymbol{s}_1)\rangle][C_{k,R}(\boldsymbol{s}_2) - \langle C_{k,R}(\boldsymbol{s}_2)\rangle]\rangle \\ &= \langle C_{k,R}(\boldsymbol{s}_1)C_{k,R}(\boldsymbol{s}_2)\rangle - \langle C_{k,R}(\boldsymbol{s}_1)\rangle\langle C_{k,R}(\boldsymbol{s}_2)\rangle. \qquad (11)\end{aligned}$$

$\Delta_{k,R}(\boldsymbol{s},\boldsymbol{s})^{1/2}$ represents the root-mean-square discrepancy to be expected between $C_{k,R}(\boldsymbol{s})$ and $\langle C_{k,R}(\boldsymbol{s})\rangle$ [7], while $\Delta_{k,R}(\boldsymbol{s}_1,\boldsymbol{s}_2)$ tells us whether the discrepancies for $\boldsymbol{s} = \boldsymbol{s}_1$ are correlated with those for $\boldsymbol{s} = \boldsymbol{s}_2$, and whether this correlation is positive or negative.

Let us note that quantities such as $\langle|\psi_k(\boldsymbol{x}_1)|^{2n}|\psi_k(\boldsymbol{x}_2)|^{2m}\rangle$ have been computed previously, but with the angle brackets representing an average over a random potential [10]. This random-potential average was subsequently shown to be equivalent to the average over the eigenfunction probability distribution $P(\psi_k|k)$ [11]. Closely related results (with random matrix theory as the starting point) were also presented in [12].

Returning to (11), we use the definition (3) of $C_{k,R}(\boldsymbol{s})$ and the combinatoric property (10) to get

$$\begin{aligned}\Delta_{k,R}(\boldsymbol{s}_1,\boldsymbol{s}_2) = \frac{A_B^2}{A_R^2}\int_R \mathrm{d}^2x_1 \int_R \mathrm{d}^2x_2\, &[\langle\psi_k(\boldsymbol{x}_1 + \tfrac{1}{2}\boldsymbol{s}_1)\psi_k(\boldsymbol{x}_2 + \tfrac{1}{2}\boldsymbol{s}_2)\rangle \\ &\times\langle\psi_k(\boldsymbol{x}_1 - \tfrac{1}{2}\boldsymbol{s}_1)\psi_k(\boldsymbol{x}_2 - \tfrac{1}{2}\boldsymbol{s}_2)\rangle \\ &+\langle\psi_k(\boldsymbol{x}_1 + \tfrac{1}{2}\boldsymbol{s}_1)\psi_k(\boldsymbol{x}_2 - \tfrac{1}{2}\boldsymbol{s}_2)\rangle\langle\psi_k(\boldsymbol{x}_1 - \tfrac{1}{2}\boldsymbol{s}_1)\psi_k(\boldsymbol{x}_2 + \tfrac{1}{2}\boldsymbol{s}_2)\rangle]. \qquad (12)\end{aligned}$$

Now using (8), we find

$$\begin{aligned}\Delta_{k,R}(\boldsymbol{s}_1,\boldsymbol{s}_2) = \frac{1}{A_R^2}\int_R \mathrm{d}^2x_1 \int_R \mathrm{d}^2x_2\, &[J_0(k|\boldsymbol{u}+\boldsymbol{s}_-|)J_0(k|\boldsymbol{u}-\boldsymbol{s}_-|) \\ &+J_0(k|\boldsymbol{u}+\boldsymbol{s}_+|)J_0(k|\boldsymbol{u}-\boldsymbol{s}_+|)] \qquad (13)\end{aligned}$$

where we have defined

$$\boldsymbol{u} = \boldsymbol{x}_1 - \boldsymbol{x}_2 \qquad \text{and} \qquad \boldsymbol{s}_\pm = \tfrac{1}{2}(\boldsymbol{s}_1 \pm \boldsymbol{s}_2). \qquad (14)$$

To proceed further, we assume that the area A_R is large, in the sense that both

$$A_R \gg \lambda^2 \qquad \text{and} \qquad A_R \gg s_{1,2}^2. \qquad (15)$$

In this case, the argument of each Bessel function is large over most of the range of the integrand, and we can use the asymptotic formula

$$J_0(x) \simeq \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{\pi}{4}\right) \qquad (16)$$

which in fact is an excellent approximation for all $x > 1$. Making the replacement (16), expanding in s/u, and keeping only those terms which are not suppressed by extra powers of either ku or s^2/u^2, we have

$$\begin{aligned}J_0(k|\boldsymbol{u}+\boldsymbol{s}|)J_0(k|\boldsymbol{u}-\boldsymbol{s}|) &\simeq \frac{2}{\pi ku}\cos\left(ku + ks\cos\theta - \frac{\pi}{4}\right)\cos\left(ku - ks\cos\theta - \frac{\pi}{4}\right) \\ &\simeq \frac{1}{\pi ku}[\sin(2ku) + \cos(2ks\cos\theta)] \qquad (17)\end{aligned}$$

where θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{s}$. We now use (17) in (13), and notice that the rapid oscillations of $\sin(2ku)$ will cause this term to integrate to zero (to a good approximation). Thus we find

$$\Delta_{k,R}(\boldsymbol{s}_1, \boldsymbol{s}_2) = \frac{1}{\pi k A_R^2} \int_R \mathrm{d}^2 x_1 \int_R \mathrm{d}^2 x_2\, u^{-1}[\cos(2ks_- \cos\theta_-) + \cos(2ks_+ \cos\theta_+)] \tag{18}$$

where θ_+ (θ_-) is the angle between $\boldsymbol{u}$ and $\boldsymbol{s}_+$ ($\boldsymbol{s}_-$).

To get a more explicit formula, we need to choose the shape of the averaging region R. For a disk of diameter d and area $A_R = \frac{1}{4}\pi d^2$, the integrals in (18) can be done in closed form by changing the integration variables to $\boldsymbol{u} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ and $\boldsymbol{v} = \boldsymbol{x}_1 + \boldsymbol{x}_2$, integrating over $\boldsymbol{v}$ subject to the constraints $|\boldsymbol{v} \pm \boldsymbol{u}| < d$, and then integrating over the magnitude of $\boldsymbol{u}$ to get

$$\Delta_{k,R}(\boldsymbol{s}_1, \boldsymbol{s}_2) = \frac{16}{3\pi^{3/2} k A_R^{1/2}} \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} [\cos(2ks_- \cos\theta) + \cos(2ks_+ \cos\theta)] \tag{19}$$

where θ_+ and θ_- have each been shifted and renamed θ. Performing the integral over θ gives us our central result,

$$\Delta_{k,R}(\boldsymbol{s}_1, \boldsymbol{s}_2) = \frac{16}{3\pi^{3/2} k A_R^{1/2}} [J_0(k|\boldsymbol{s}_1 - \boldsymbol{s}_2|) + J_0(k|\boldsymbol{s}_1 + \boldsymbol{s}_2|)]. \tag{20}$$

We now turn to a study of the implications of (20). The expected discrepancy between $C_{k,R}(\boldsymbol{s})$ and $\langle C_{k,R}(\boldsymbol{s})\rangle$ is given by

$$\Delta_{k,R}(\boldsymbol{s}, \boldsymbol{s})^{1/2} = 1.38(k^2 A_R)^{-1/4}[\tfrac{1}{2} + \tfrac{1}{2} J_0(2ks)]^{1/2} \tag{21}$$

where the function in square brackets attains its maximum value of one when $ks = 0$. Since (21) is proportional to $k^{-1/2}$, it is $O(\hbar^{1/2})$; thus, the RMS amplitude of the expected fluctuations in $C_{k,R}(\boldsymbol{s})$ vanishes in the classical limit. However, this amplitude is not numerically small unless $A_R \gg \lambda^2$. Both of these points are in accord with Berry's original (qualitative) discussion of the approach of $C_{k,R}(\boldsymbol{s})$ to $\langle C_{k,R}(\boldsymbol{s})\rangle$ as $\hbar \to 0$ [1]. Furthermore, $\Delta_{k,R}(\boldsymbol{s}, \boldsymbol{s})^{1/2}$ is the same order in $\hbar$ as any corrections due to scars. This is consistent with the idea [3] that scars represent a particular organization of the Gaussian fluctuations in the eigenfunction, rather than constituting an additional phenomenon.

For comparison, we turn to the numerical results of Li and Robnik [5] for the Robnik billiard [6]. We computed $C_{k,R}(\boldsymbol{s})$ using the eigenfunction with $k = 790.644$, shown in figure 3 of [5], which was kindly supplied to us by Li and Robnik. The averaging region R was taken to be a disk with diameter $d = 0.273 = 34.4\lambda$. For this value of d, the coefficient of the bracketed function in (21) is 0.100. The leading corrections to (20) from terms we have neglected (due to our various approximations) are suppressed by an extra factor of either $1/kA_R^{1/2} = 0.005$ or s^2/A_R; for $ks = 30$, $s^2/A_R = 0.025$. In figures 1–4, we plot the actual correlation function $C_{k,R}(\boldsymbol{s})$ as a full curve, along with a shaded band encompassing the range $\langle C_{k,R}(\boldsymbol{s})\rangle \pm \Delta_{k,R}(\boldsymbol{s}, \boldsymbol{s})^{1/2}$, for four different averaging regions. In these plots, $\boldsymbol{s}$ is taken to be parallel to the x-axis; we found visually similar results (which are not shown) for other directions of $\boldsymbol{s}$. We see that the actual $C_{k,R}(\boldsymbol{s})$ usually lies within the shaded band, but also has (sometimes large) excursions outside it. Without attempting a detailed quantitative analysis, we can say that these graphs are qualitatively consistent with what we expect.

Li and Robnik [5] suggested that the discrepancy between $C_{k,R}(\boldsymbol{s})$ and $\langle C_{k,R}(\boldsymbol{s})\rangle$ could be reduced by averaging $C_{k,R}(\boldsymbol{s})$ over the direction of $\boldsymbol{s}$. Let us define

$$\bar{C}_{k,R}(s) \equiv \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} C_{k,R}(\boldsymbol{s}(\phi)) \tag{22}$$

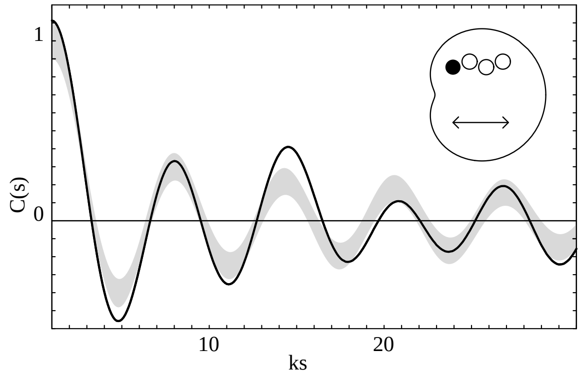

Figure 1. The autocorrelation function $C_{k,R}(\boldsymbol{s})$ is shown as a full curve. The grey band depicts $\langle C_{k,R}(\boldsymbol{s})\rangle \pm \Delta_{k,R}(\boldsymbol{s},\boldsymbol{s})^{1/2}$, which is the expected root-mean-square range of $C_{k,R}(\boldsymbol{s})$. The Robnik billiard is shown in the inset; the averaging region R is indicated by the full circle, and the direction of the separation vector $\boldsymbol{s}$ is indicated by the direction of the two-headed arrow, which has unit length, equal to 126 wavelengths.

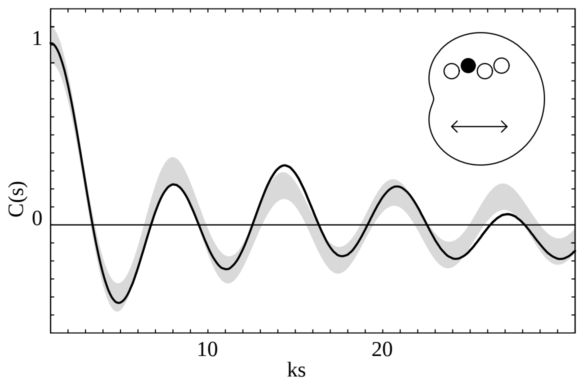

Figure 2. Same as figure 1.

where $\boldsymbol{s}(\phi) = (s\cos\phi, s\sin\phi)$. Obviously, we have

$$\langle \bar{C}_{k,R}(s)\rangle = \langle C_{k,R}(\boldsymbol{s})\rangle = J_0(ks). \tag{23}$$

We also define

$$\begin{aligned}\bar{\Delta}_{k,R}(s_1,s_2) &\equiv \langle \bar{C}_{k,R}(s_1)\bar{C}_{k,R}(s_2)\rangle - \langle \bar{C}_{k,R}(s_1)\rangle\langle \bar{C}_{k,R}(s_2)\rangle \\ &= \int_0^{2\pi}\frac{\mathrm{d}\phi_1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\phi_2}{2\pi}\,\Delta_{k,R}(\boldsymbol{s}_1(\phi_1),\boldsymbol{s}_2(\phi_2)).\end{aligned} \tag{24}$$

Then, using (20), we find

$$\bar{\Delta}_{k,R}(s_1,s_2) = \frac{32}{3\pi^{3/2}kA_R^{1/2}}\int_0^{2\pi}\frac{\mathrm{d}\phi}{2\pi}\,J_0(k[s_1^2+s_2^2\pm 2s_1s_2\cos\phi]^{1/2}) \tag{25}$$

where $\phi = \phi_1 - \phi_2$; this integral can be performed to yield

$$\bar{\Delta}_{k,R}(s_1,s_2) = \frac{32}{3\pi^{3/2}kA_R^{1/2}}\,J_0(ks_1)J_0(ks_2). \tag{26}$$

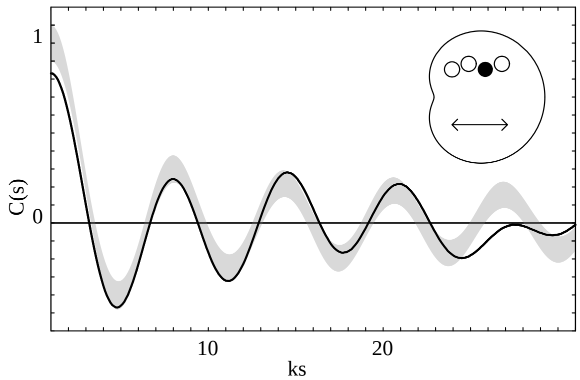

Figure 3. Same as figure 1.

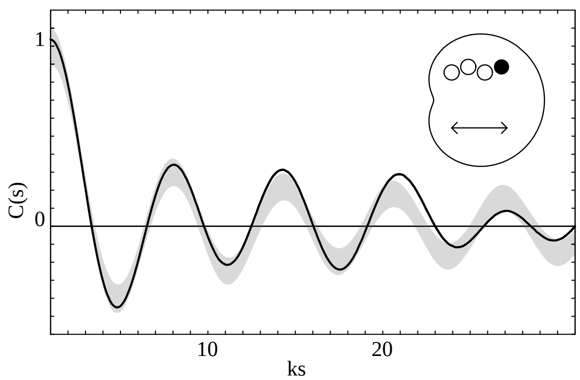

Figure 4. Same as figure 1.

The fact that $\bar{\Delta}_{k,R}(s_1, s_2)$ is proportional to $\langle\bar{C}_{k,R}(s_1)\rangle\langle\bar{C}_{k,R}(s_2)\rangle$ has dramatic consequences; it implies that $\bar{C}_{k,R}(s)/\langle\bar{C}_{k,R}(s)\rangle$ must be independent of s. To demonstrate this, we choose a set of orthonormal basis functions $f_n(s)$, $n = 0, 1, 2, \ldots$, with $f_0(s)$ chosen to be equal to $\langle\bar{C}_{k,R}(s)\rangle$. We require orthonormality in the sense that

$$\int_0^\infty \mathrm{d}s\, w(s) f_n(s) f_m(s) = \delta_{nm} \tag{27}$$

where $w(s)$ is any weight function which ensures the convergence and correct normalization of the integral when $n = m = 0$. (Since $\langle\bar{C}_{k,R}(s)\rangle = J_0(ks)$, we could construct such a set of basis functions by starting with the Bessel functions $J_n(ks)$ and then performing Gram–Schmidt orthogonalization.) Once we have the basis functions, we can write

$$\begin{aligned} \bar{C}_{k,R}(s) &= \sum_{n=0}^{\infty} c_n f_n(s) \\ c_n &= \int_0^\infty \mathrm{d}s\, w(s) f_n(s) \bar{C}_{k,R}(s) \end{aligned} \tag{28}$$

where the c_n's should be regarded as random variables. By construction, we have

$$\langle c_n\rangle = \delta_{n0}. \tag{29}$$

Using (26), we can also compute the expected value of c_n^2. We find

$$\begin{aligned}\langle c_n^2\rangle - \langle c_n\rangle^2 &= \int_0^\infty \mathrm{d}s_1\, w(s_1)\, f_n(s_1) \int_0^\infty \mathrm{d}s_2\, w(s_2)\, f_n(s_2)\, \bar{\Delta}_{k,R}(s_1, s_2)\\ &= \frac{32}{3\pi^{3/2} k A_R^{1/2}} \int_0^\infty \mathrm{d}s_1\, w(s_1)\, f_n(s_1)\, f_0(s_1) \int_0^\infty \mathrm{d}s_2\, w(s_2)\, f_n(s_2)\, f_0(s_2)\\ &= \frac{32}{3\pi^{3/2} k A_R^{1/2}} \delta_{n0}. \end{aligned} \tag{30}$$

Thus $\langle c_n^2\rangle = 0$ if $n \neq 0$, indicating that the probability distribution has no support for any nonzero c_n other than c_0. Therefore $\bar{C}_{k,R}(s) \propto f_0(s) = \langle \bar{C}_{k,R}(s)\rangle$.

However, we must remember that there are additional contributions to $\bar{\Delta}_{k,R}(s_1, s_2)$ which are suppressed by an extra factor of either $1/kA_R^{1/2}$ or s^2/A_R, and that these will make small corrections to the functional form of $\bar{\Delta}_{k,R}(s_1, s_2)$. This means that $\bar{C}_{k,R}(s)/\langle \bar{C}_{k,R}(s)\rangle$ should be independent of s up to corrections of order $1/kA_R^{1/2}$ and s^2/A_R.

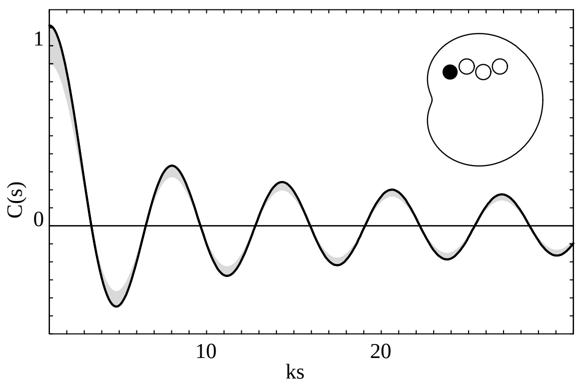

Figure 5. The autocorrelation function $\bar{C}_{k,R}(s)$, averaged over the direction of the separation vector $\boldsymbol{s}$, is shown as a full curve. The grey band depicts $\langle \bar{C}_{k,R}(s)\rangle \pm \bar{\Delta}_{k,R}(s, s)^{1/2}$, which is the expected root-mean-square range of $\bar{C}_{k,R}(s)$. The Robnik billiard is shown in the inset; the averaging region R is indicated by the full circle.

The discrepancy between $\bar{C}_{k,R}(s)$ and $\langle \bar{C}_{k,R}(s)\rangle$ is governed by $\bar{\Delta}_{k,R}(s, s)^{1/2}$. In figure 5, we plot the actual direction-averaged correlation function $\bar{C}_{k,R}(s)$ as a full curve, along with a shaded band encompassing the range $\langle \bar{C}_{k,R}(s)\rangle \pm \bar{\Delta}_{k,R}(s, s)^{1/2}$, for one of the four averaging regions used previously. We see that the actual $\bar{C}_{k,R}(s)$ is consistent with our expectations. In figure 6, we plot $[\bar{C}_{k,R}(s) - \bar{C}_{k,R}(0) J_0(ks)] + \bar{C}_{k,R}(0)$ for all four averaging regions; this quantity should be independent of s and equal to $\bar{C}_{k,R}(0)$, up to corrections of order $1/kA_R^{1/2} = 0.005$ and $s^2/A_R = 0.025$ (for $ks = 30$). (We plot this quantity instead of the ratio $\bar{C}_{k,R}(s)/J_0(ks)$ because the latter is dominated by numerical errors near the zeros of its denominator.) The plots are remarkably flat; the small glitches which are present are most likely due to the build-up of round-off errors in the numerical computation. These plots confirm our prediction that $\bar{C}_{k,R}(s)/\langle \bar{C}_{k,R}(s)\rangle$ should be independent of s.

This concludes our analysis of the fluctuations in the autocorrelation function for the case of a circular averaging region. We now consider the dependence of $\Delta_{k,R}(\boldsymbol{s}_1, \boldsymbol{s}_2)$ on the shape of the averaging region R.

For noncircular R, the integrals in (18) cannot be done in closed form for nonzero $\boldsymbol{s}_{1,2}$.

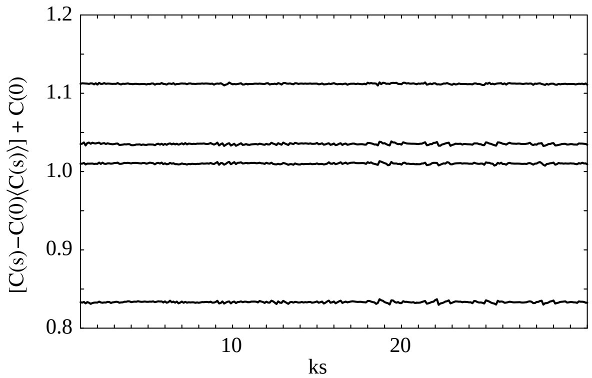

Figure 6. $[\bar{C}_{k,R}(s) - \bar{C}_{k,R}(0)J_0(ks)] + \bar{C}_{k,R}(0)$ for each of the four averaging regions, with an expanded vertical scale. This quantity is predicted to be independent of s.

For the special case $s_1 = s_2 = 0$, however, we can evaluate (18) for a rectangular averaging region. This will give us the shape dependence of $\Delta_{k,R}(\mathbf{0},\mathbf{0})$, and therefore (we hope) some idea of the shape dependence of $\Delta_{k,R}(s_1, s_2)$ for general s_1 and s_2. For a rectangle with edge lengths a and b, each of which is much greater than the wavelength $\lambda = 2\pi/k$, we find

$$\Delta_{k,R}(\mathbf{0},\mathbf{0}) = \frac{4}{3\pi k A_R^{1/2}}[g(\xi) + g(\xi^{-1}) - (\xi + \xi^{-1})^{3/2}] \tag{31}$$

where $A_R = ab$ is the rectangle's area, $\xi = b/a$ is the ratio of edge lengths, and

$$g(\xi) = \xi^{-3/2} + 3\xi^{-1/2}\sinh^{-1}\xi. \tag{32}$$

The ratio of $\Delta_{k,R}(\mathbf{0},\mathbf{0})$ for a rectangle to $\Delta_{k,R}(\mathbf{0},\mathbf{0})$ for a circle of the same area decreases monotonically from 0.99 for $\xi = 1$ to 0.74 for $\xi = 10$, and ultimately falls off as $\xi^{-1/2}\ln\xi$ for large ξ. Thus, for moderate values of ξ, $\Delta_{k,R}(\mathbf{0},\mathbf{0})$ exhibits only a mild shape dependence.

Earlier numerical computations of $C_{k,R}(s)$ [4, 5, 13] are all in qualitative agreement with our considerations; specifically, the average discrepancy between $C_{k,R}(s)$ and $\langle C_{k,R}(s)\rangle$ is always roughly given by $(k^2 A_R)^{-1/4}$. A detailed comparison is hindered by two issues. First, in accord with Berry's original definition, earlier authors usually work with an autocorrelation function $\tilde{C}_{k,R}(s)$ which, in our notation, is

$$\tilde{C}_{k,R}(s) = \frac{C_{k,R}(s)}{C_{k,R}(\mathbf{0})}. \tag{33}$$

From our point of view, $\tilde{C}_{k,R}(s)$ is a much more complicated object than $C_{k,R}(s)$; there is no simple expression for $\langle\tilde{C}_{k,R}(s)\rangle$, because the relevant functional integral (of $C_{k,R}(s)/C_{k,R}(\mathbf{0})$ times the probability distribution $P(\psi_k|k)$ over ψ_k) cannot be done using the simple Gaussian combinatorics of (10). Of course, by definition $\tilde{C}_{k,R}(\mathbf{0}) = 1$, and so essentially what happens is that any fluctuation in $C_{k,R}(\mathbf{0})$ shows up as a multiplicative enhancement or suppression of fluctuations in $\tilde{C}_{k,R}(s)$ at nonzero s.

Another problem occurs if an axis of symmetry of the billiard passes through the averaging region. Every energy eigenfunction is either symmetric or antisymmetric under reflection about such an axis; this can be handled analytically by writing

$$\psi_k(x, y) = \frac{1}{\sqrt{2}}[\chi_k(x, y) + \chi_k(x, -y)] \tag{34}$$

(where we have illustratively assumed an eigenfunction symmetric about the x-axis), and treating $\chi_k(\boldsymbol{x})$ as a Gaussian random variable. This approach considerably enhances the complexity of the analysis, however; for example, the number of independent terms on the right-hand side of (10) grows from three to 48. The simplest solution is to do numerical analysis with averaging regions that do not cross any axes of symmetry, as we have done here.

Finally, we note that all of our results have a straightforward generalization to higher dimensions. For a D-dimensional billiard, the autocorrelation function becomes [1]

$$\begin{aligned}\langle C_{k,R}(\boldsymbol{s})\rangle &= \frac{\int \mathrm{d}^D p\, \delta(\boldsymbol{p}^2 - k^2)\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{s}}}{\int \mathrm{d}^D p\, \delta(\boldsymbol{p}^2 - k^2)} \\ &= 2^{(D-2)/2}\Gamma(D/2)\frac{J_{(D-2)/2}(ks)}{(ks)^{(D-2)/2}} \\ &\equiv F_D(ks)\end{aligned} \tag{35}$$

where $J_\nu(x)$ is a Bessel function. The generalizations of (20) and (26), which follow from the properties of $F_D(ks)$, are

$$\Delta_{k,R}(\boldsymbol{s}_1, \boldsymbol{s}_2) = \frac{\gamma_D}{k^{D-1}V_R^{(D-1)/D}}[F_D(k|\boldsymbol{s}_1 - \boldsymbol{s}_2|) + F_D(k|\boldsymbol{s}_1 + \boldsymbol{s}_2|)] \tag{36}$$

and

$$\bar{\Delta}_{k,R}(s_1, s_2) = \frac{2\gamma_D}{k^{D-1}V_R^{(D-1)/D}}F_D(ks_1)F_D(ks_2) \tag{37}$$

where V_R is the D-dimensional volume of the spherical averaging region, and γ_D is a numerical factor which we have not computed.

To conclude, we have performed an analysis of the autocorrelation function $C_{k,R}(\boldsymbol{s})$ under the assumption that the energy eigenfunction $\psi_k(\boldsymbol{x})$ behaves like a Gaussian random variable, in a sense which we have made precise. We find that, for a two-dimensional billiard, $C_{k,R}(\boldsymbol{s})$ should have $\mathrm{O}(\hbar^{1/2})$ fluctuations about its expected value $\langle C_{k,R}(\boldsymbol{s})\rangle = J_0(ks)$; scars from isolated periodic orbits would give corrections to $C_{k,R}(\boldsymbol{s})$ which are also $\mathrm{O}(\hbar^{1/2})$. We have given analytic formulae for the root-mean-square amplitude of the expected fluctuations in $C_{k,R}(\boldsymbol{s})$. We find that a particularly useful object to study is $\bar{C}_{k,R}(s)$, which is $C_{k,R}(\boldsymbol{s})$ averaged over the angle of $\boldsymbol{s}$. We predict that $\bar{C}_{k,R}(s)/J_0(ks)$ is independent of s, a prediction which is very well satisfied by the numerical results of Li and Robnik for the Robnik billiard.

Acknowledgments

We are grateful to Baowen Li and Marko Robnik for kindly providing us with some of their data. We thank Eric Heller and Steve Tomsovic for helpful discussions. This work was supported in part by NSF Grant PHY-91-16964.

References

[1] Berry M V 1977 *J. Phys. A: Math. Gen.* **10** 2083
[2] Heller E J 1984 *Phys. Rev. Lett.* **53** 1515
[3] O'Connor P, Gehlen J and Heller E J 1987 *Phys. Rev. Lett.* **58** 1296
[4] MacDonald S W and Kaufman A N 1988 *Phys. Rev.* A **37** 3067

[5] Li B and Robnik M 1994 *J. Phys. A: Math. Gen.* **27** 5509

[6] Robnik M 1983 *J. Phys. A: Math. Gen.* **16** 3971; 1984 *J. Phys. A: Math. Gen.* **17** 1049

[7] Srednicki M 1994 *Phys. Rev.* E **50** 888

[8] Eckhardt B, Fishman S, Keating J, Agam O, Main J and Müller K 1995 *Phys. Rev.* E **52** 5893

[9] Ramond P 1990 *Field Theory: A Modern Primer* (Redwood City, CA: Addison-Wesley)

[10] Prigodin V N 1995 *Phys. Rev. Lett.* **74** 1566
Prigodin V N, Taniguchi N, Kudrolli A, Kidambi V and Sridhar S 1995 *Phys. Rev. Lett.* **75** 2392; erratum, in press

[11] Srednicki M 1996 *Phys. Rev.* E **54** 954

[12] Alhassid Y and Lewenkopf C H 1995 *Phys. Rev. Lett.* **75** 3922

[13] Aurich R and Steiner F 1993 *Physica* **64D** 185